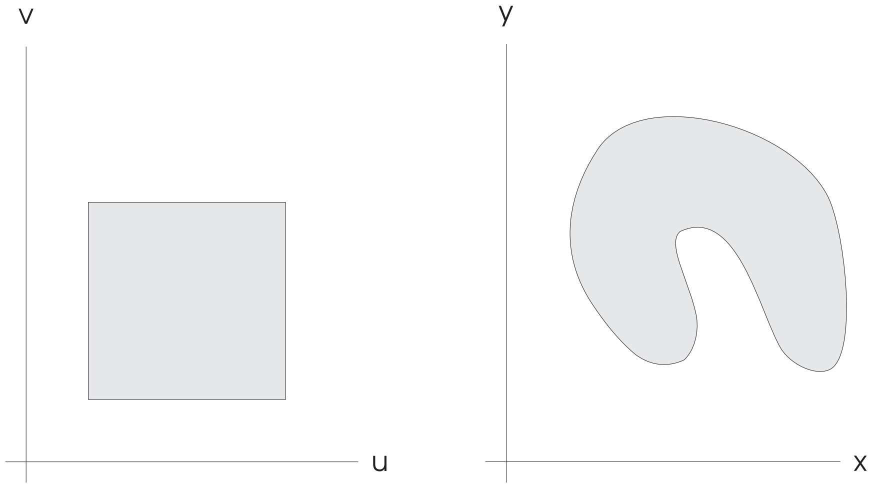
8-31-1993

Coordinate transformations.

A **coordinate transformation** of the plane is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. I'll usually assume that f has continuous partial derivatives, and that f is "essentially" one-to-one in the region of interest. (A function is **one-to-one** if different inputs produce different outputs.)

A coordinate transformation will usually be given by an equation $(x, y) = f(u, v)$. You can think of it as deforming or moving things in the u-v plane and placing them in the x-y plane.



I'm going to look at some important special cases.

1. **Linear maps.** A **linear map** $\mathbb{R}^2 \to \mathbb{R}^2$ has the form

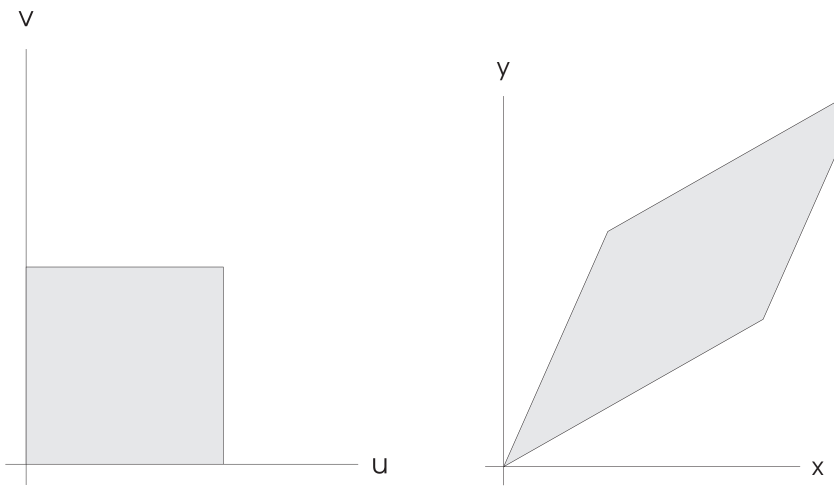
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

a, b, c, and d are numbers. In equation form, this is

$$x = au + bv, \quad y = cu + dv.$$

Linear maps always leave the origin fixed, since $u = 0$, $v = 0$ gives $x = 0$, $y = 0$. And as you might expect, linear maps carry lines to lines.

Moreover, a linear map takes squares to parallelograms:

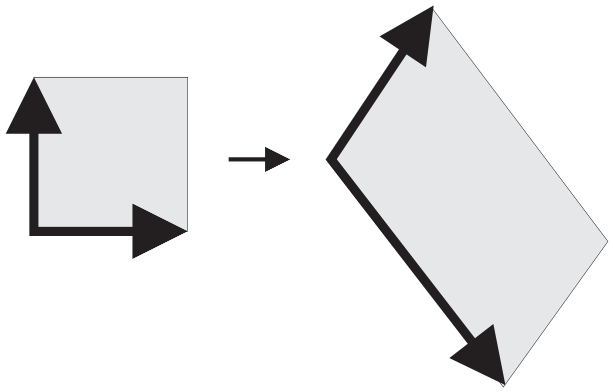


For example, look at what happens to the unit square $0 \le u \le 1$, $0 \le v \le 1$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

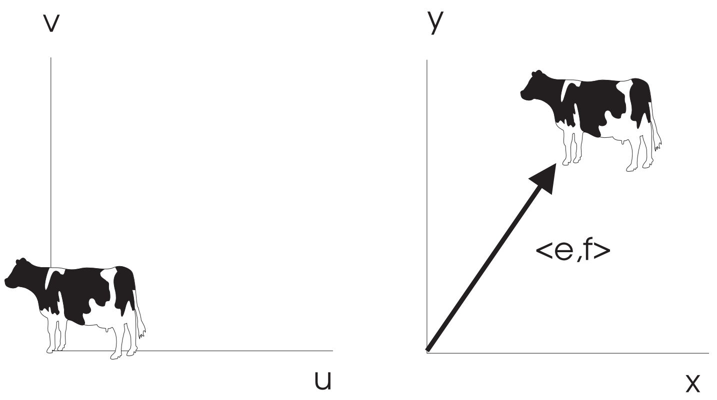
The vectors $\langle 1,0\rangle$, $\langle 0,1\rangle$ determine the unit square. They are mapped to the vector $\langle a,c\rangle$, $\langle b,d\rangle$, two adjacent sides of the parallelogram.



2. **Translations.** A **translation** $\mathbb{R}^2 \to \mathbb{R}^2$ has the form

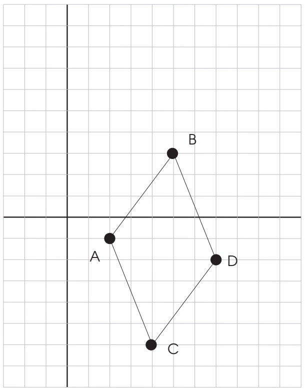
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

e and f are numbers. This translation just translates everything by the vector $\langle e,f\rangle$.

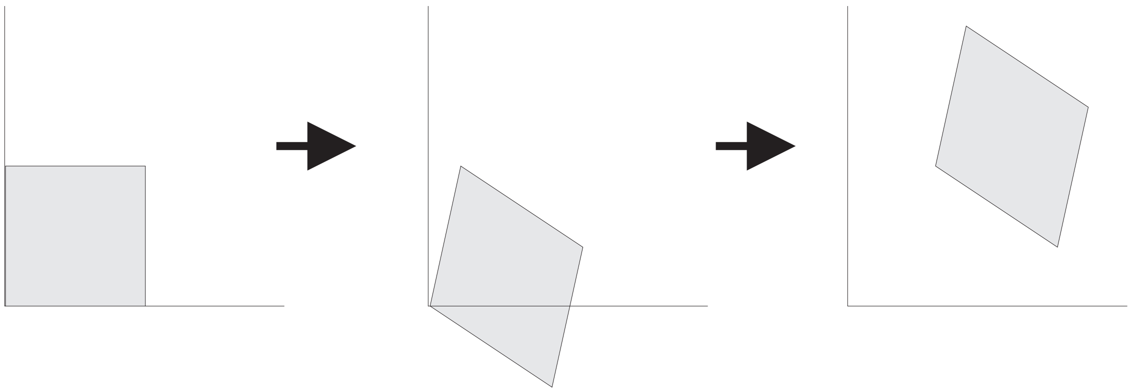


Example. Find a linear map which carries the unit square $0 \le u \le 1$, $0 \le v \le 1$ to the parallelogram in the x-y plane with vertices $A(2,-1)$, $B(5,3)$, $C(4,-6)$, $D(7,-2)$.

I'm going to make a rough sketch of the parallelogram first. I want to know the orientation of the vertices — e.g. that B and C are next to A, and that D is opposite A.



I'm going to do the transformation in two steps. First, I'll take the square to a parallelogram which is the right size and shape, but which has a corner at the origin. Next, I'll move the parallelogram so it's at the right place.



The vectors from A to B and to C are $\overrightarrow{AB} = \langle 3, 4\rangle$ and $\overrightarrow{AC} = \langle 2, -5\rangle$. I saw above that if I construct a matrix with $\langle 3, 4\rangle$ as the first column and $\langle 2, -5\rangle$ as the second column, then it will multiply $\langle 1, 0\rangle$ to $\langle 3, 4\rangle$ and $\langle 0, 1\rangle$ to $\langle 2, -5\rangle$:

$$\begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

So the following transformation takes the unit square to a parallelogram of the right shape:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This transformation takes $(0, 0)$ to $(0, 0)$; I want $(0, 0)$ to go to the point A (which I used as the base point for my two vectors). To fix this, just translate by A:

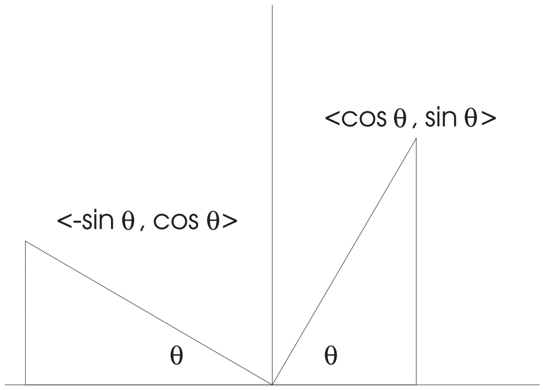
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \quad \square$$

3. **Rotations.** A **rotation** counterclockwise through an angle θ is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

To see this, just check that the unit vectors $\langle 1,0\rangle$, $\langle 0,1\rangle$ go to the right places:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

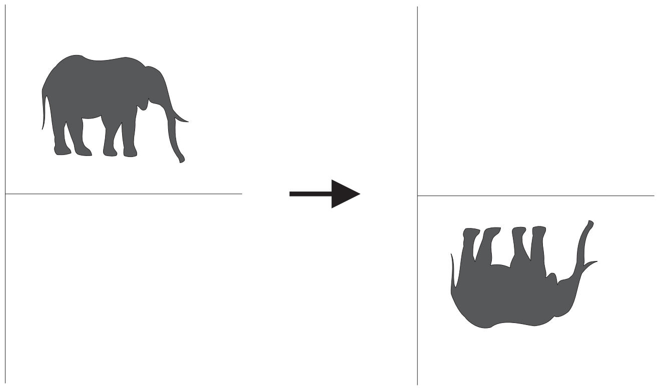


You can see from the picture that $\langle 1,0\rangle$ and $\langle 0,1\rangle$ have both been rotated by an angle θ. Other vectors can be built out of these two vectors, so other vectors are rotated by θ as well.

4. **Reflections.** This is how to do a **reflection** across the x-axis:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ -v \end{bmatrix}.$$

It is clear that this reflects things across the x-axis, because it simply negates the second component.



What about reflection across a line L making an angle θ with the origin? It's messy to do using analytic geometry, but very easy using matrices. Simply do a rotation through $-\theta$ which carries L to the $x-axis$. Next, reflect across the x-axis. Finally, do a rotation through $-\theta$ which carries the x-axis back to L. Each of these transformations can be accomplished by matrix multiplication; just multiply the three matrices to do reflection across L.

Translations, rotations, and reflections are examples of **rigid motions**. They preserve distances between points, as well as areas.